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A NOTE ON PSEUDOSYMMETRIC KENMOTSU MANIFOLDS SATISFYING CERTAIN CONDITIONS

P. Uygun and M. Atçeken
Department of Mathematics
Faculty of Arts and Sciences
Aksaray University, 68100
Aksaray, Turkey
pakizeuygun@hotmail.com

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Abstract

The purpose of this paper is to study pseudosymmetric and Ricci pseudosymmetric of a Kenmotsu manifolds. We obtain the necessary and sufficient conditions for a Kenmotsu manifold, W_2 pseudosymmetric, W_2 Ricci pseudosymmetric, W_3 pseudosymmetric and W_3 Ricci pseudosymmetric. Finally, some interesting results on Kenmotsu manifolds are obtained.

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1. Introduction

In a Riemannian manifold, the Riemannian curvature tensor is R and for each $\beta_1, \beta_2 \in \chi(M)$, if $R(\beta_1, \beta_2) \cdot R = 0$, then the manifold is said to be semisymmetric. Similarly, if $R(\beta_1, \beta_2) \cdot S = 0$, the manifold is called Ricci semisymmetric, if $R(\beta_1, \beta_2) \cdot P = 0$, the manifold is called projective semisymmetric, $R(\beta_1, \beta_2) \cdot \widetilde{Z} = 0$ and the manifold is called concircular semisymmetric, where S is the Ricci curvature tensor, P is the projective curvature tensor and $\widetilde{Z}$ is the concircular curvature tensor. Studies on the symmetric Riemannian manifolds started with Cartan [5]. In the following periods, many authors have studied the curvature tensors of various manifolds [4, 6, 10, 14, 15].

In [13] is studied on a classification of almost $C(a)-$manifolds. T. Mert researched pseudosymmetric and Ricci pseudosymmetric almost $C(a)-$manifolds. For an almost $C(a)-$manifold, Riemannian pseudosymmetric, Riemannian Ricci pseudosymmetric, Ricci pseudosymmetric, projective pseudossymmetric, projective Ricci pseudosymmetric, concircular pseudosymmetric and concircular Ricci pseudosymmetric cases are considered and new results are obtained.

K. Kenmotsu researched a class of a contact Riemannian manifold and call them Kenmotsu manifold [12]. He studied that if Kenmotsu manifold satisfies the condition $R(\beta_1, \beta_2).R = 0$, then the manifold is of negative curvature -1, where R is the Riemannian curvature tensor of type $(1,3)$ and $R(\beta_1, \beta_2)$ denotes the derivation of the tensor algebra at each point of the tangent space. The properties of Kenmotsu manifolds have been studied by several authors[7, 8, 19, 21, 22].

In this article, we have researched the pseudosymmetric and Ricci pseudosymmetric of Kenmotsu manifold. For Kenmotsu manifold, W_2 pseudosymmetric, W_2 Ricci pseudosymmetric, W_3 pseudosymmetric and W_3 Ricci pseudosymmetric cases are considered. Then some results are obtained and classifications have been made.

2. Preliminaries

Let M be a $(2n+1)-$dimensional almost contact metric manifold with an almost contact metric structure (ϕ, ξ, η, g), that is, ϕ is an $(1,1)$ tensor field, ξ is a vector field, η is a 1-form and the Riemanniann metric g on M satisfy

$$\phi^2(\beta_1) = -\beta_1 + \eta(\beta_1)\xi, \quad \eta(\phi\beta_1) = 0, \tag{2.1}$$

$$\eta(\xi) = 1, \quad \phi\xi = 0, \quad \eta(\phi) = 0 \tag{2.2}$$

for all $\beta_1, \beta_2 \in \chi(M)$ [16]. Let g be Riemannian metric compatible with (ϕ, ξ, η), that is

$$g(\phi\beta_1, \phi\beta_2) = g(\beta_1, \beta_2) - \eta(\beta_1)\eta(\beta_2), \tag{2.3}$$

or equivalently,

$$g(\beta_1, \phi\beta_2) = -g(\phi\beta_1, \beta_2) \quad \text{and} \quad g(\beta_1, \xi) = \eta(\beta_1) \tag{2.4}$$

for all $\beta_1, \beta_2 \in \chi(M)$ [2]. If in addition to above relation

$$(\nabla_{\beta_1}\phi)\beta_2 = -\eta(\beta_2)\phi\beta_1 - g(\beta_1, \phi\beta_2)\xi, \tag{2.5}$$

and

$$\nabla_{\beta_1}\xi = \beta_1 - \eta(\beta_1)\xi, \tag{2.6}$$

where ∇ denotes the Riemannian connection of g hold, then $M(\phi, \xi, \eta, g)$ is called Kenmotsu manifold. Kenmotsu manifold becomes a Kenmotsu manifold if

$$g(\beta_1, \phi\beta_2) = d\eta(\beta_1, \beta_2). \tag{2.7}$$

In a Kenmotsu manifold M, the following relation holds [12, 8]:

$$(\nabla_{\beta_1}\eta)\beta_2 = g(\beta_1, \beta_2) - \eta(\beta_1)\eta(\beta_2), \tag{2.8}$$

$$R(\beta_1, \beta_2)\xi = \eta(\beta_1)\beta_2 - \eta(\beta_2)\beta_1, \tag{2.9}$$

$$R(\xi, \beta_1)\beta_2 = \eta(\beta_2)\beta_1 - g(\beta_1, \beta_2)\xi, \tag{2.10}$$

$$S(\beta_1, \xi) = -2n\eta(\beta_1), \tag{2.11}$$

$$Q\xi = -2n\xi, \tag{2.12}$$

where R is the Riemannian curvature tensor and S is Ricci tensor defined by $S(\beta_1, \beta_2) = g(Q\beta_1, \beta_2)$, where Q is Ricci operator. It yields to

$$S(\phi\beta_1, \phi\beta_2) = S(\beta_1, \beta_2) + 2n\eta(\beta_1)\eta(\beta_2). \tag{2.13}$$

Definition 2.1. A Kenmotsu manifold M is said to be an $\eta-$Einstein manifold if its Ricci tensor S of the form

$$S(\beta_1, \beta_2) = \alpha g(\beta_1, \beta_2) + \beta\eta(\beta_1)\eta(\beta_2) \tag{2.14}$$

for arbitrary vector fields β_1, β_2; where α and β are functions on (M^{2n+1}, g). If $\beta = 0$, then $\eta-$ Einstein manifold becomes Einstein manifold [3].

On a semi-Riemannian manifold (M,g), for a $(0,k)-$type tensor field $(0,k)$-type tensor field T and $(0,2)$-type tensor field A, $(0,k+2)$-type Tachibana tensor field $Q(A,T)$ is defined as

$$\begin{aligned} Q(A,T)(\beta_{11},\beta_{12},...,\beta_{1k};\beta_1,\beta_2) &= -T((\beta_1 \wedge_A \beta_2)\beta_{11},\beta_{12},...,\beta_{1k}) \\ &- T(\beta_{11},(\beta_1 \wedge_A \beta_2)\beta_1,\beta_{13},...,\beta_{1k}) \\ &\cdot \\ &\cdot \\ &\cdot \\ &- T(\beta_{11},\beta_{12},...,(\beta_1 \wedge_A \beta_2)\beta_{1k}), \end{aligned} \tag{2.15}$$

for all $\beta_{11},\beta_{12},...,\beta_{1k},\beta_1,\beta_2 \in \chi(M)$, where

$$(\beta_1 \wedge_A \beta_2)\beta_3 = A(\beta_2,\beta_3)\beta_1 - A(\beta_1,\beta_3)\beta_2. \tag{2.16}$$

for all $\beta_1,\beta_2,\beta_3 \in \chi(M)$.

3. Characterization of Pseudosymmetric Kenmotsu manifold

In this section, the case of pseudosymmetry and Ricci pseudosymmetry of Kenmotsu manifold are investigated. According to W_2 curvature tensor and W_3 curvature tensor, the pseudosymmetrical and Ricci pseudosymmetrical cases of the Kenmotsu manifold can be given as follows.

Let (M,g) be an $(2n+1)$-dimensional Riemannian manifold. Then the W_2 curvature tensor is defined by [18]. Furthermore, W_2 the curvature tensor for Riemannian manifold (M^{2n+1},g) is given by

$$W_2(\beta_1,\beta_2)\beta_3 = R(\beta_1,\beta_2)\beta_3 - \frac{1}{2n}[g(\beta_2,\beta_3)Q\beta_1 - g(\beta_1,\beta_3)Q\beta_2] \tag{3.1}$$

for all $\beta_1,\beta_2,\beta_3 \in \chi(M)$ [18]. If we choose, respectively, $\beta_1 = \xi$ and $\beta_3 = \xi$ in (3.1), then we obtain as follows:

$$W_2(\xi,\beta_2)\beta_3 = \eta(\beta_3)\beta_2 + \frac{1}{2n}\eta(\beta_3)Q\beta_2, \tag{3.2}$$

$$W_2(\beta_1,\beta_2)\xi = \eta(\beta_1)\beta_2 - \eta(\beta_2)\beta_1 - \frac{1}{2n}(\eta(\beta_2)Q\beta_1 - \eta(\beta_1)Q\beta_2). \tag{3.3}$$

In addition, we choose $\beta_3 = \xi$ in (3.2), we obtain as follows:

$$W_2(\xi,\beta_2)\xi = \beta_2 + \frac{1}{2n}Q\beta_2. \tag{3.4}$$

Definition 3.1. Let M be Kenmotsu manifold with $(2n+1)-$dimensional, R be the Riemannian curvature tensor of M, S be the Ricci curvature tensor of M and W_2 be the W_2 curvature tensor.

(i) If the pair $R \cdot W_2$ and $Q(g, W_2)$ are linearly dependent, that is, if a λ_1 function can be found on the set
$M_1 = \{\beta_1 \in M | g(\beta_1) \neq W_2(\beta_1)\}$ such that

$$R.W_2 = \lambda_1 Q(g, W_2), \tag{3.5}$$

the M manifold is called a W_2 pseudosymmetric manifold.

(ii) If the pair $R \cdot W_2$ and $Q(S, W_2)$ are linearly dependent, that is, if a λ_2 function can be found on the set
$M_2 = \{\beta_1 \in M | S(\beta_1) \neq W_2(\beta_1)\}$ such that

$$R.W_2 = \lambda_2 Q(S, W_2), \tag{3.6}$$

the M manifold is called a W_2 Ricci pseudosymmetric manifold. Particularly, if $\lambda_1 = 0$, then this manifold is said to be semisymmetric [9].

Let us now investigate the cases of W_2 pseudosymmetry and W_2 Ricci pseudosymmetry.

Theorem 3.2. *If a $(2n+1)-$dimensional M Kenmotsu manifold is a W_2 pseudosymmetric manifold, then M is an Einstein manifold.*

Proof. Let us suppose that Kenmotsu manifold M is a W_2 pseudosymmetric manifold. Then, we can write

$$(R(\beta_1, \beta_2) \cdot W_2)(\beta_3, \beta_4, \beta_5) = \lambda_1 Q(g, W_2)(\beta_5, \beta_4, \beta_3; \beta_1, \beta_2), \tag{3.7}$$

for each $\beta_1, \beta_2, \beta_3, \beta_4, \beta_5 \in \chi(M)$. This means that

$$\begin{aligned} & R(\beta_1, \beta_2) W_2(\beta_3, \beta_4)\beta_5 - W_2(R(\beta_1, \beta_2)\beta_3, \beta_4)\beta_5 - W_2(\beta_3, R(\beta_1, \beta_2)\beta_4)\beta_5 \\ & -W_2(\beta_3, \beta_4) R(\beta_1, \beta_2)\beta_5 \\ = \; & -\lambda_1 \{W_2((\beta_1 \wedge_g \beta_2)\beta_5, \beta_4)\beta_3 + W_2(\beta_5, (\beta_1 \wedge_g \beta_2)\beta_4)\beta_3 \\ & +W_2(\beta_5, \beta_4)(\beta_1 \wedge_g \beta_2)\beta_3\}. \end{aligned} \tag{3.8}$$

Here taking $\beta_1 = \beta_3 = \xi$ and by using (2.9), (2.10) and (2.16) in (3.8), we reach at

$$\begin{aligned} & \eta(W_2(\xi, \beta_4)\beta_5)\beta_2 - g(\beta_2, W_2(\xi, \beta_4)\beta_5) - R(\beta_2 - \eta(\beta_2)\xi, \beta_4)\beta_5 \\ & -W_2(\xi, \eta(\beta_4)\beta_2 - g(\beta_4, \beta_2)\xi)\beta_5 - \eta(W_2(\xi, \beta_2)\beta_5)\beta_4 + g(\beta_4, R(\xi, \beta_2)\beta_5) \\ = \; & -\lambda_1 \{W_2(g(\beta_2, \beta_5)\xi - \eta(\beta_5)\beta_2, \beta_4)\xi + W_2(\beta_5, g(\beta_2, \beta_4)\xi - \eta(\beta_4)\beta_2) \\ & +W_2(\beta_5, \beta_4)(\eta(\beta_2)\xi - \beta_2)\}. \end{aligned} \tag{3.9}$$

If we use (3.2), (3.3) and taking $\beta_5 = \xi$ in (3.9) and make the necessary abbreviations, then we have

$$\begin{aligned}
&\eta(\beta_4 + \frac{Q\beta_4}{2n})\beta_2 - g(\beta_2, \beta_4 + \frac{Q\beta_4}{2n})\xi - W_2(\beta_2, \beta_4)\xi + \eta(\beta_2)W_2(\xi, \beta_4)\xi \\
&-\eta(\beta_4)W_2(\xi, \beta_2)\xi - W_2(\xi, \beta_4)\beta_2 + \eta(\beta_2)W_2(\xi, \beta_4)\xi \\
= \; &-\lambda_1\{\eta(\beta_2)(\beta_4 + \frac{Q\beta_4}{2n}) - (\eta(\beta_2)\beta_4 - \eta(\beta_4)\beta_2 - \eta(\beta_4)\frac{Q\beta_2}{2n} \\
&+\eta(\beta_2)\frac{Q\beta_4}{2n}) - \eta(\beta_4)(\beta_2 + \frac{Q\beta_2}{2n}) + \eta(\beta_2)(\beta_4 + \frac{Q\beta_4}{2n}) \\
&-W_2(\xi, \beta_4)\beta_2\}. \qquad (3.10)
\end{aligned}$$

Taking the inner product with $\xi \in \chi(M)$ on both sides of (3.10) and make use of (3.2), then we can infer

$$S(\beta_4, \beta_2) = -2ng(\beta_4, \beta_2).$$

This completes our proof. □

Theorem 3.3. *If a $(2n+1)-$dimensional M Kenmotsu manifold is a W_2 Ricci pseudosymmetric manifold, then M is an Einstein manifold.*

Proof. Let us assume that Kenmotsu manifold M is a W_2 Ricci pseudosymmetric manifold. This implies that

$$(R(\beta_1, \beta_2) \cdot W_2)(\beta_3, \beta_4, \beta_5) = \lambda_2 Q(S, W_2)(\beta_5, \beta_4, \beta_3; \beta_1, \beta_2), \qquad (3.11)$$

for each $\beta_1, \beta_2, \beta_3, \beta_4, \beta_5 \in \chi(M)$, that is,

$$\begin{aligned}
&R(\beta_1, \beta_2)W_2(\beta_3, \beta_4)\beta_5 - W_2(R(\beta_1, \beta_2)\beta_3, \beta_4)\beta_5 - W_2(\beta_3, R(\beta_1, \beta_2)\beta_4)\beta_5 \\
&-W_2(\beta_3, \beta_4)R(\beta_1, \beta_2)\beta_5 \\
= \; &-\lambda_2\{W_2((\beta_1 \wedge_S \beta_2)\beta_5, \beta_4)\beta_3 + W_2(\beta_5, (\beta_1 \wedge_S \beta_2)\beta_4)\beta_3 \\
&+W_2(\beta_5, \beta_4)(\beta_1 \wedge_S \beta_2)\beta_3\}. \qquad (3.12)
\end{aligned}$$

Here, taking $\beta_1 = \beta_3 = \xi$ and using (2.9), (2.16) in (3.12), we arrive at

$$\begin{aligned}
&\eta(W_2(\xi, \beta_4)\beta_5)\beta_2 - g(\beta_2, W_2(\xi, \beta_4)\beta_5) - W_2(\beta_2 - \eta(\beta_2)\xi, \beta_4)\beta_5 \\
&-W_2(\xi, \eta(\beta_4)\beta_2 - g(\beta_4, \beta_2)\xi)\beta_5 - \eta(W_2(\xi, \beta_2)\beta_5)\beta_4 + g(\beta_4, W_2(\xi, \beta_2)\beta_5) \\
= \; &-\lambda_2\{W_2(S(\beta_2, \beta_5)\xi + 2n\eta(\beta_5)\beta_2, \beta_4)\xi + W_2(\beta_5, S(\beta_2, \beta_4)\xi \\
&+2n\eta(\beta_4)\beta_2)\xi + 2nW_2(\beta_5, \beta_4)(\beta_2 - \eta(\beta_2)\xi)\}. \qquad (3.13)
\end{aligned}$$

If ξ is taken of β_5 at (3.13), considering (2.11), (3.2), then we get

$$\begin{aligned}
&\eta(\beta_4+\frac{Q\beta_4}{2n})\beta_2-g(\beta_2,\beta_4+\frac{Q\beta_4}{2n})\xi-W_2(\beta_2,\beta_4)\xi+\eta(\beta_2)W_2(\xi,\beta_4)\xi\\
&-\eta(\beta_4)W_2(\xi,\beta_2)\xi-W_2(\xi,\beta_4)\beta_2+\eta(\beta_2)W_2(\xi,\beta_4)\xi\\
=\;&-\lambda_2\{-4n\eta(\beta_2)(\beta_4+\frac{Q\beta_4}{2n})+2n(\eta(\beta_2)\beta_4-\eta(\beta_4)\beta_2-\eta(\beta_4)\frac{Q\beta_2}{2n}\\
&+\eta(\beta_2)\frac{Q\beta_4}{2n})+2n\eta(\beta_4)(\beta_2+\frac{Q\beta_2}{2n})+2nW_2(\xi,\beta_4)\beta_2\}. \qquad (3.14)
\end{aligned}$$

Inner product both sides of (3.14) by $\xi\in\chi(M)$ and make the necessary adjustments, we obtain

$$S(\beta_4,\beta_2)=-2ng(\beta_4,\beta_2). \qquad (3.15)$$

This proves our assertion. □

Let (M,g) be an $(2n+1)$-dimensional Riemannian manifold. Then the W_3 curvature tensor is defined by [18]. Furthermore, W_3 the curvature tensor for Riemannian manifold (M^{2n+1},g) is given by

$$W_3(\beta_1,\beta_2)\beta_3=R(\beta_1,\beta_2)\beta_3-\frac{1}{2n}[S(\beta_1,\beta_3)\beta_2-g(\beta_2,\beta_3)Q\beta_1] \qquad (3.16)$$

for all $\beta_1,\beta_2,\beta_3\in\chi(M)$ [18]. If we choose, respectively, $\beta_1=\xi$ and $\beta_3=\xi$ in (3.16), then we obtain as follows:

$$W_3(\xi,\beta_2)\beta_3=2\eta(\beta_3)\beta_2-g(\beta_2,\beta_3)\xi, \qquad (3.17)$$

$$W_3(\beta_1,\beta_2)\xi=2\eta(\beta_1)\beta_2-\eta(\beta_2)\beta_1+\eta(\beta_2)\frac{Q\beta_1}{2n}. \qquad (3.18)$$

In addition, we choose $\beta_3=\xi$ in (3.17), we obtain as follows:

$$W_3(\xi,\beta_2)\xi=2\beta_2-2\eta(\beta_2)\xi. \qquad (3.19)$$

Definition 3.4. Let M be Kenmotsu manifold with $(2n+1)-$dimensional, R be the Riemannian curvature tensor of M, S be the Ricci curvature tensor of M and W_3 be the W_3-curvature tensor.

(i) If the pair $R\cdot W_3$ and $Q(g,W_3)$ are linearly dependent, that is, if a λ_3 function can be found on the set

$M_3=\{\beta_1\in M|g(\beta_1)\neq W_3(\beta_1)\}$ such that

$$R.W_3=\lambda_3Q(g,W_3), \qquad (3.20)$$

the M manifold is called a W_3 pseudosymmetric manifold.

(ii) If the pair $R\cdot W_3$ and $Q(S,W_3)$ are linearly dependent, that is, if a λ_4 function can be found on the set

$M_4=\{\beta_1\in M|S(\beta_1)\neq W_3(\beta_1)\}$ such that

$$R.W_3=\lambda_4Q(S,W_3), \qquad (3.21)$$

the M manifold is called a W_3 Ricci pseudosymmetric manifold [9].

Let us now investigate the cases of W_3 pseudosymmetry and W_3 Ricci pseudosymmetry.

Theorem 3.5. *If a $(2n+1)-$dimensional M Kenmotsu manifold is a W_3 pseudosymmetric manifold, then M is a semisymmetric manifold.*

Proof. Let us suppose that Kenmotsu manifold M is a W_3 pseudosymmetric manifold. Then, we can write

$$(R(\beta_1,\beta_2)\cdot W_3)(\beta_3,\beta_4,\beta_5) = \lambda_3 Q(g,W_3)(\beta_5,\beta_4,\beta_3;\beta_1,\beta_2), \tag{3.22}$$

for each $\beta_1,\beta_2,\beta_3,\beta_4,\beta_5 \in \chi(M)$. This means that

$$\begin{aligned} & R(\beta_1,\beta_2)W_3(\beta_3,\beta_4)\beta_5 - W_3(R(\beta_1,\beta_2)\beta_3,\beta_4)\beta_5 - W_3(\beta_3,R(\beta_1,\beta_2)\beta_4)\beta_5 \\ & -W_3(\beta_3,\beta_4)R(\beta_1,\beta_2)\beta_5 \\ = \; & -\lambda_3\{W_3((\beta_1\wedge_g\beta_2)\beta_5,\beta_4)\beta_3 + W_3(\beta_5,(\beta_1\wedge_g\beta_2)\beta_4)\beta_3 \\ & +W_3(\beta_5,\beta_4)(\beta_1\wedge_g\beta_2)\beta_3\}, \end{aligned} \tag{3.23}$$

that is, in the last equality taking $\beta_1=\beta_3=\xi$ and using (2.9), (2.10) and (3.17) in (3.23), we obtain

$$\begin{aligned} & \eta(W_3(\xi,\beta_4)\beta_5)\beta_2 - g(\beta_2,W_3(\xi,\beta_4)\beta_5) - W_3(\beta_2-\eta(\beta_2)\xi,\beta_4)\beta_5 \\ & -W_3(\xi,\eta(\beta_4)\beta_2 - g(\beta_4,\beta_2)\xi)\beta_5 - \eta(W_3(\xi,\beta_2)\beta_5)\beta_4 + g(\beta_4,W_3(\xi,\beta_2)\beta_5) \\ = \; & -\lambda_3\{W_3(g(\beta_2,\beta_5)\xi-\eta(\beta_5)\beta_2,\beta_4)\xi + W_3(\beta_5,g(\beta_2,\beta_4)\xi-\eta(\beta_4)\beta_2) \\ & +W_3(\beta_5,\beta_4)(\eta(\beta_2)\xi-\beta_2)\}. \end{aligned} \tag{3.24}$$

If we use (3.17), (3.18) and taking $\beta_5=\xi$ in (3.24) and make the necessary abbreviations, then we have

$$\begin{aligned} & \eta(2\beta_4-2\eta(\beta_4)\xi)\beta_2 - g(\beta_2,2\beta_4-2\eta(\beta_4)\xi)\xi - W_3(\beta_2,\beta_4)\xi \\ & +\eta(\beta_2)W_3(\xi,\beta_4)\xi - \eta(\beta_4)W_3(\xi,\beta_2)\xi - W_3(\xi,\beta_4)\beta_2 \\ & +\eta(\beta_2)W_3(\xi,\beta_4)\xi \\ = \; & -\lambda_3\{\eta(\beta_2)(2\beta_4-2\eta(\beta_4)\xi) - (2\eta(\beta_2)\beta_4-\eta(\beta_4)\beta_2 \\ & +\eta(\beta_4)\frac{Q\beta_2}{2n}) - \eta(\beta_4)(2\beta_2-2\eta(\beta_2)\xi) + \eta(\beta_2)(2\beta_4 \\ & -2\eta(\beta_4)\xi) - (2\eta(\beta_2)\beta_4 - 2g(\beta_4,\beta_2)\xi)\}. \end{aligned} \tag{3.25}$$

Taking the inner product with $\xi\in\chi(M)$ on both sides of (3.25) and make use of (3.19), then we arrive

$$2\lambda_3\left[g(\beta_4,\beta_2)-\eta(\beta_4)\eta(\beta_2)\right]=0. \tag{3.26}$$

On the other hand, we know that from (3.26) and (2.3), we conclude

$$2\lambda_3 g(\phi\beta_4,\phi\beta_2)=0.$$

This completes our proof. □

Theorem 3.6. *If a $(2n+1)-$dimensional M Kenmotsu manifold is a W_3 Ricci pseudosymmetric manifold, then M is a semisymmetric manifold.*

Proof. Let us assume that Kenmotsu manifold M is a W_3 Ricci pseudosymmetric manifold. This implies that

$$(R(\beta_1,\beta_2)\cdot W_3)(\beta_3,\beta_4,\beta_5) = \lambda_4 Q(S,W_3)(\beta_5,\beta_4,\beta_3;\beta_1,\beta_2), \tag{3.27}$$

for each $\beta_1,\beta_2,\beta_3,\beta_4,\beta_5 \in \chi(M)$, that is,

$$\begin{aligned} & R(\beta_1,\beta_2)W_3(\beta_3,\beta_4)\beta_5 - W_3(R(\beta_1,\beta_2)\beta_3,\beta_4)\beta_5 - W_3(\beta_3,R(\beta_1,\beta_2)\beta_4)\beta_5 \\ & -W_3(\beta_3,\beta_4)R(\beta_1,\beta_2)\beta_5 \\ = \; & -\lambda_4\{W_3((\beta_1\wedge_S\beta_2)\beta_5,\beta_4)\beta_3 + W_3(\beta_5,(\beta_1\wedge_S\beta_2)\beta_4)\beta_3 \\ & +W_3(\beta_5,\beta_4)(\beta_1\wedge_S\beta_2)\beta_3\}. \end{aligned} \tag{3.28}$$

Taking $\beta_1=\beta_3=\xi$ and using (2.9), (3.17) in (3.28), we have

$$\begin{aligned} & \eta(W_3(\xi,\beta_4)\beta_5)\beta_2 - g(\beta_2,W_3(\xi,\beta_4)\beta_5) - W_3(\beta_2-\eta(\beta_2)\xi,\beta_4)\beta_5 \\ & -W_3(\xi,\eta(\beta_4)\beta_2 - g(\beta_4,\beta_2)\xi)\beta_5 - \eta(W_3(\xi,\beta_2)\beta_5)\beta_4 + g(\beta_4,W_3(\xi,\beta_2)\beta_5) \\ = \; & -\lambda_4\{W_3(S(\beta_2,\beta_5)\xi + 2n\eta(\beta_5)\beta_2,\beta_4)\xi + W_3(\beta_5,S(\beta_2,\beta_4)\xi \\ & +2n\eta(\beta_4)\beta_2)\xi + 2nW_3(\beta_5,\beta_4)(\beta_2-\eta(\beta_2)\xi)\}. \end{aligned} \tag{3.29}$$

If we use (3.17), (3.19) and setting $\beta_5=\xi$ in (3.29), then we get

$$\begin{aligned} & \eta(2\beta_4-2\eta(\beta_4)\xi)\beta_2 - g(\beta_2,2\beta_4-2\eta(\beta_4)\xi)\xi - W_3(\beta_2,\beta_4)\xi \\ & +\eta(\beta_2)W_3(\xi,\beta_4)\xi - \eta(\beta_4)W_3(\xi,\beta_2)\xi - W_3(\xi,\beta_4)\beta_2 \\ & +\eta(\beta_2)W_3(\xi,\beta_4)\xi \\ = \; & -\lambda_4\{-4n\eta(\beta_2)(2\beta_4-2\eta(\beta_4)\xi) + 2n(2\eta(\beta_2)\beta_4 - \eta(\beta_4)\beta_2 \\ & +\eta(\beta_4)\frac{Q\beta_2}{2n}) + 2n(2\eta(\beta_2)\beta_4 - 2g(\beta_4,\beta_2)\xi) \\ & +2n\eta(\beta_4)(2\beta_2-2\eta(\beta_2)\xi)\}. \end{aligned} \tag{3.30}$$

Inner product both sides of (3.30) by $\xi\in\chi(M)$, we obtain

$$4n\lambda_4\left[g(\beta_4,\beta_2)-\eta(\beta_4)\eta(\beta_2)\right]=0.$$

Consequently, it follows from (2.3)

$$4n\lambda_4 g(\phi\beta_4,\phi\beta_2)=0,$$

which proves our assertion. □

4. CONCLUSION

The aim of this paper is to study pseudosymmetric and Ricci pseudosymmetric of a Kenmotsu manifolds. We obtain the necessary and sufficient conditions for a Kenmotsu manifold, W_2 pseudosymmetric, W_2 Ricci pseudosymmetric, W_3 pseudosymmetric and W_3 Ricci pseudosymmetric.

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